JOURNAL OF ALGEBRA 160, 524–551 (1993)

The Classification of Certain Butler Groups

PAUL HILL*

Department of Algebra, Combinatorics and Analysis, Auburn University, Auburn, Alabama 36849

AND

CHARLES MEGIBBEN†

Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37235

Communicated by Barbara L. Osofsky

Received January 26, 1990

1. INTRODUCTION

Unlike finitely generated abelian groups, those of finite rank continue to defy a satisfactory description of their structure in terms of numerical or other useful invariants. The difficulty, of course, is with the torsion-free case, that is, with subgroups of finite dimensional rational vector spaces. Thus these relatively small torsion-free groups present a serious obstacle to the expansion of the rapidly developing classification theory of abelian groups; see, for example, [H1, HM1, R]. Hereafter, all groups will be assumed to be abelian. Although at this time there appears to be no way to treat successfully finite rank torsion-free groups in general, some progress is being made toward the classification of those finite rank torsion-free groups known as Butler groups. *Butler groups* are simply the pure subgroups of completely decomposable groups of finite rank [A, AV1, B]. Richman [R] classified a comparatively small family of Butler groups that he called "doubly incomparable." In [AV3] Arnold and Vinsonhaler carried this classification (up to quasi-isomorphism) further to include those finite rank strongly indecomposable groups that occur as pure subgroups of corank 1 in completely decomposable torsion-free groups. And concurrently with our investigation, Arnold and Vinsonhaler [AV5] have

* Supported in part by National Science Foundation Grant DMS-8521770.
† Supported in part by National Science Foundation Grant DMS-8420900.

0021-8693/93 $5.00

Copyright © 1993 by Academic Press, Inc.
All rights of reproduction in any form reserved.

now classified these groups up to isomorphism, which is also one of the results of the present paper.

As suggested above, our objective in this paper is to develop the classification theory of Butler groups beyond its present state, but we hasten to add that the general problem of classifying all Butler groups remains open. Perhaps the general problem is unreasonable, but we believe that a more nearly complete solution than we have given here will be obtained in due course.

For the most part, we shall follow the standard notation and terminology laid down in [F]. There are, however, a few slight departures, all of which have precedents in the work of other researchers in the field. For example, by a height we mean a sequence $s=(s_p)_{p\in \mathbf{P}}$ where $\mathbf{P}$ denotes the set of rational primes and each s_p is either a nonnegative integer or the symbol ∞. When x is an element of the torsion-free group G, we indicate its height by $|x|=(|x|_p)_{p\in \mathbf{P}}$ where $|x|_p$ denotes the ordinary p-height of x as computed in G. Heights are, of course, ordered pointwise; and with each height s and each torsion-free group G, we associate the fully-invariant subgroup $G(s)=\{x\in G : |x|\geqslant s\}$. As in customary, two heights s and t are said to be equivalent provided (1) $s_p=t_p$ for all but finitely many primes p and (2) $s_p=\infty$ if and only if $t_p=\infty$. With each height s, we also associate the fully-invariant subgroup

$$G(s^*)=\langle x\in G(s) : |x| \text{ is not equivalent to } s\rangle.$$

Equivalence classes of heights are called *types* and will be denoted by Greek letters. If G is a torsion-free group and σ is a type, then $G(\sigma)=\bigcup_{s\in\sigma} G(s)$ is a subgroup that is both pure and fully-invariant. On the other hand, the subgroup $G(\sigma^*)=\bigcup_{s\in\sigma} G(s^*)$ is not necessarily pure in G. If, however, G is a Butler group with the property that $G(\sigma^*)$ is pure in G for all types σ, then we call G a B_0-*group* [A]. Heights and types form, in the obvious manner, distributive lattices, and we shall indicate the lattice operations by the standard symbols $\wedge$ and $\vee$.

The unifying point of view of the present paper is the application of equivalence theorems. Such theorems have already had an impact on the elucidation of the structure of isotype subgroups of simply presented abelian groups [HM1, HM3, L], a class of groups that contains all Butler groups since the torsion-free simply presented groups are precisely the completely decomposable ones. Two subgroups H and H' of the group G are said to be equivalent if there is an automorphism ψ of G such that $\psi(H)=H'$. By an *equivalence theorem*, we mean any theorem that asserts under appropriate hypotheses that two subgroups of a given group are equivalent. Usually such theorems are proved in a slightly more general form; namely, H and H' are subgroups of G and G', respectively, and an isomorphism $\psi: G \to G'$ is constructed such that $\psi(H) = H'$. Under these circumstances, $G/H \cong G'/H'$, and frequently one can construct $\psi: G \to G'$ so as to induce a prescribed isomorphism $\phi: G/H \to G'/H'$ in the sense that $\phi(x+H) = \psi(x) + H'$ for all $x \in G$. Clearly, one possibility of proving that two groups are isomorphic is to imbed them in a common group in such a way that they can be shown to be equivalent. This proves more than is required, but we cite [W, H2] and Theorem 3.11 below as examples of the efficacy of this strategy.

An obvious necessary condition for an isomorphism $\phi: G/H \to G'/H'$ to be inducible by an isomorphism $\psi: G \to G'$ is that ϕ *respects heights* in the sense that

$$\phi((G(s)+H)/H) = (G'(s)+H')/H' \qquad \text{for all heights } s. \tag{1}$$

LEMMA 1.1. *If $\phi: G/H \to G'/H'$ is an isomorphism that respects heights, then*

(i) $\phi((G(s^*)+H)/H) = (G'(s^*)+H')/H'$,

(ii) $\phi((G(\sigma)+H)/H) = (G'(\sigma)+H')/H'$,

(iii) $\phi((G(\sigma^*)+H)/H) = (G'(\sigma^*)+H')/H'$.

Moreover, if ϕ respects heights, then for each type σ there is an induced isomorphism

$$\phi_\sigma : \frac{G(\sigma)/G(\sigma^*)}{(H \cap G(\sigma) + G(\sigma^*))/G(\sigma^*)} \to \frac{G'(\sigma)/G'(\sigma^*)}{(H' \cap G'(\sigma) + G'(\sigma^*))/G'(\sigma^*)}.$$

Proof. (i), (ii), and (iii) are straightforward consequences of ϕ respecting heights. The final assertion follows from (ii), (iii), and the natural isomorphisms

$$\frac{G(\sigma)}{H \cap G(\sigma) + G(\sigma^*)} \cong \frac{G(\sigma)+H}{G(\sigma^*)+H} \qquad \text{and}$$

$$\frac{G'(\sigma)}{H' \cap G'(\sigma) + G'(\sigma^*)} \cong \frac{G'(\sigma)+H'}{G'(\sigma^*)+H'}.$$

We are thus led inevitably to the following problem: If H and H' are pure subgroups of the completely decomposable torsion-free groups G and G', respectively, and if $\phi: G/H \to G'/H'$ is an isomorphism that respects heights, then find useful necessary and sufficient conditions for ϕ to be induced by an isomorphism $\psi: G \to G'$. At the present time, the next theorem constitutes the best available solution to this problem.

THEOREM 1.2. *Suppose H and H' are pure subgroups of the completely decomposable torsion-free groups G and G', respectively, and that $\phi: G/H \to G'/H'$ is an isomorphism that respects heights. Then a necessary and sufficient condition for ϕ to be induced by some isomorphism $\psi: G \to G'$ is that, for each type σ, the map*

$$\phi_\sigma : \frac{G(\sigma)/G(\sigma^*)}{(H(\sigma)+G(\sigma^*))/G(\sigma^*)} \to \frac{G'(\sigma)/G'(\sigma^*)}{(H'(\sigma)+G'(\sigma^*))/G'(\sigma^*)}$$

be induced by an isomorphism from $G(\sigma)/G(\sigma^*)$ to $G'(\sigma)/G'(\sigma^*)$.

The proof of this theorem, which may be found in [HM4], depends on the interplay between various concepts first introduced in [HM2]. Clearly, a necessarily condition for ϕ_σ to be induced by an isomorphism from $G(\sigma)/G(\sigma^*)$ to $G'(\sigma)/G'(\sigma^*)$ is that the groups $(H(\sigma)+G(\sigma^*))/G(\sigma^*) \cong H(\sigma)/H \cap G(\sigma^*)$ and $(H'(\sigma)+G'(\sigma^*))/G'(\sigma^*) \cong H'(\sigma)/H' \cap G'(\sigma^*)$ be isomorphic. But since these groups are completely decomposable and homogeneous of type σ [F, Theorem 86.6], isomorphism reduces to

$$\operatorname{rank}(H(\sigma)/H \cap G(\sigma^*)) = \operatorname{rank}(H'(\sigma)/H' \cap G'(\sigma^*)). \tag{2}$$

As it turns out, requiring (2) for all types σ, will not suffice to insure that ϕ is induced by an isomorphism between G and G'. In fact there exists a pure subgroup H of a finite rank completely decomposable group G and an isomorphism $\phi: G/H \to G/H$ that respects heights, but such that no automorphism of G induces ϕ [HM4]. On the other hand, if we require the quotient groups $G(\sigma)/(H(\sigma)+G(\sigma^*))$ and $G'(\sigma)/(H'(\sigma)+G'(\sigma^*))$ to be torsion-free, then no such example can exist. In other words, if we require a more stringent form of purity on the subgroups H and H', then condition (2) holding for all σ will be adequate to insure that each height respecting isomorphism $\phi: G/H \to G'/H'$ is induced by some appropriate isomorphism $\psi: G \to G'$.

DEFINITION 1.3. A pure subgroup H of torsion-free group G is said to be weakly $*$-pure provided

$$H \cap [G(s^*) + pG(s)] = H \cap G(s^*) + pH(s)$$

for all heights s and all primes p.

A proof of the following elementary result may be found in [HM4].

LEMMA 1.4. *Let H be a pure subgroup of the torsion-free group G. If the quotient group $G(\sigma)/(H(\sigma)+G(\sigma^*))$ is torsion-free for all types σ, then H is weakly $*$-pure in G. Conversely, if H is weakly $*$-pure in G and $G(\sigma)/G(\sigma^*)$ is torsion-free, then $G(\sigma)/(H(\sigma)+G(\sigma^*))$ is also torsion-free.*

For weakly $*$-pure subgroups of completely decomposable torsion-free groups, we can obtain an equivalence theorem analogous to those available in the case of local groups [HM3]. Indeed this follows from Theorem 1.2 and the appropriate equivalence theorem for pure subgroups of completely decomposable homogeneous groups [HM4].

THEOREM 1.5. *Suppose that H and H' are weakly $*$-pure subgroups of the completely decomposable torsion-free groups G and G', respectively. Then there is an isomorphism $\psi: G \to G'$ such that $\psi(H) = H'$ if and only if the following two conditions are satisfied:*

(1) *there is an isomorphism $\phi: G/H \to G'/H'$ that respects heights,*

(2) *for all types σ,* $\operatorname{rank}(H(\sigma)/(H \cap G(\sigma^*))) = \operatorname{rank}(H'(\sigma)/(H' \cap G'(\sigma^*)))$.

COROLLARY 1.6. *Suppose H and H' are pure subgroups of the completely decomposable torsion-free groups G and G', respectively, such that $H(\sigma) \subseteq G'(\sigma^*)$ for all types σ. Then $\psi(H) = H'$ for some isomorphism $\psi: G \to G'$ if and only if there exists an isomorphism $\phi: G/H \to G'/H'$ that respects heights.*

Proof. The condition $H(\sigma) \subseteq G(\sigma^*)$ implies that $G(\sigma)/(H(\sigma) + G(\sigma^*)) = G(\sigma)/G(\sigma^*)$ and rank $(H(\sigma)/H \cap G(\sigma^*)) = 0$.

The terminology introduced in Definition 1.3 derives from [HM2] wherein a pure subgroup H of the torsion-free group is said to be $*$-pure provided that $H \cap G(s^*) = H(s^*)$ and $H \cap [G(s^*) + pG(s)] = H \cap G(s^*) + pH(s)$ for all heights s and all primes p. But a $*$-pure subgroup of a finite rank completely decomposable group is necessarily a direct summand [DR], and hence $*$-purity is too strong a condition to be useful in the study of Butler groups. On the other hand, weakly $*$-pure subgroups of completely decomposable groups are relevant to Butler groups as is demonstrated in our next result.

PROPOSITION 1.7. *Let H be a pure subgroup of the finite rank completely decomposable group*

(i) *If the types of the homogeneous components of G are pairwise incomparable, then H is weakly $*$-pure in G.*

(ii) *If H is a balanced subgroup of G, then a necessary and sufficient condition for H to be weakly $*$-pure is that G/H be a B_0-group.*

Proof. (i) The significance of the additional hypothesis on G is that $G(\sigma^*) = 0$ when G has a nonzero σ-homogeneous component and $G(\sigma) = G(\sigma^*)$ otherwise. In the first instance, $G(\sigma)/(H(\sigma) + G(\sigma^*)) = G(\sigma)/H(\sigma)$ is torsion-free since $H(\sigma)$ is pure in G; while $G(\sigma)/(H(\sigma)+G(\sigma^*))=0$ whenever $G(\sigma)=G(\sigma^*)$.

(ii) If H is balanced in G, then $(G/H)(\sigma)/(G/H)(\sigma^*)$ is canonically isomorphic to $G(\sigma)/(H(\sigma)+G(\sigma^*))$. Thus the condition follows immediately from Lemma 1.4 and the definition of B_0-groups.

In Sections 2 and 4 below, we shall consider some fairly routine consequences of Theorem 1.5, all of which can be established by ad hoc methods without resort to this theorem. These results certainly yield less insight into the structure of the groups under consideration than do applications of the comparable equivalence theorems for local groups [HM1, HM3], but this may be due as much to the underdeveloped status of the subject as to the inherent intractability of torsion-free groups. On the other hand, in Section 3 we obtain via a specialized equivalence theorem a result (Theorem 3.11) of greater depth.

2. Butler Groups as Pure Subgroups

As we mentioned earlier, Butler groups can be defined as being the pure subgroups of completely decomposable groups of finite rank. A fundamental result of Butler [B] establishes the fact that these groups are precisely the torsion-free quotients of completely decomposable groups of finite rank. In this section, we investigate Butler groups as pure subgroups, whereas in Section 4 we study Butler groups as quotients. We begin with a simple application of Theorem 1.5.

Theorem 2.1. *Two pure subgroups G and G' of a finite rank completely decomposable group A, with incomparable homogeneous components, are equivalent if and only if there exists an isomorphism $\phi\colon A/G \to A/G'$ between the corresponding quotients that respects heights.*

Proof. The expression "with incomparable homogeneous components" is taken here as a convenient abbreviation for the more precise formulation in Proposition 1.7(i). Thus, by that proposition, both G and G' are weakly $*$-pure subgroups of A. It remains only to verify condition (2) of Theorem 1.5. For those types σ for which $A(\sigma)=A(\sigma^*)$, we have

$$\operatorname{rank}\left(\frac{G(\sigma)}{G\cap A(\sigma^*)}\right)=0=\operatorname{rank}\left(\frac{G'(\sigma)}{G'\cap A(\sigma^*)}\right).$$

When $A(\sigma^*)=0$, we have by [F, Lemma 86.8] direct decompositions $A(\sigma)=G(\sigma)\oplus B$ and $A(\sigma)=G'(\sigma)\oplus B'$. But then the existence of an isomorphism $\phi\colon A/G\to A/G'$ that respects heights implies, via Lemma 1.1(ii), that $B \cong A(\sigma)/G(\sigma) \cong A(\sigma)/G'(\sigma) \cong B'$. Consequently, by finiteness of rank, we also have

$$\operatorname{rank}\left(\frac{G(\sigma)}{G \cap A(\sigma^*)}\right) = \operatorname{rank} G(\sigma) = \operatorname{rank} G'(\sigma) = \operatorname{rank}\left(\frac{G'(\sigma)}{G' \cap A(\sigma^*)}\right)$$

for those types σ with $A(\sigma^*) = 0$.

The following corollary is of independent interest.

COROLLARY 2.2. *Let A be a completely decomposable group of finite rank whose homogeneous components are incomparable. Any two balanced subgroups B and B' are isomorphic provided that the corresponding quotients A/B and A/B' are isomorphic.*

Let G be a subgroup of an arbitrary abelian group A. By the height spectrum of a coset $a+G$ we mean the collection of heights of the various elements of the coset $a+G$. The height spectrum of $a+G$ is denoted by $\|a+G\|$, so

$$\|a+G\| = \{|a+g| : g \in G\}.$$

Observe that if G and G' are subgroups of A for which there is an isomorphism $\phi: A/G \to A/G'$ that preserves height spectrums (in the sense that $\|a+G\| = \|\phi(a+G)\|$ for all $a \in A$), then ϕ respects heights. Therefore the following corollary is an immediate consequence of Theorem 2.1.

COROLLARY 2.3. *Two Butler groups G and G' are isomorphic provided there exist isomorphic completely decomposable groups A and A', with incomparable homogeneous components, that contain G and G' as pure subgroups, respectively, and admit an isomorphism $\phi: A/G \to A'/G'$ that preserves height spectrums.*

In connection with the preceding corollary, we remark that it may be quite difficult to determine whether or not an isomorphism $\phi: A/G \to A/G'$ preserves height spectrums, but the question simplifies significantly when A/G and A/G' have rank 1. Thus, for the remainder of this section, we restrict ourselves to Butler groups G having corank 1 in a completely decomposable group A.

Suppose that $A = A_1 \oplus \cdots \oplus A_n$ is a completely decomposable group where each A_i is a nonzero subgroup of $\mathbb{Q}$, the additive group of the rational numbers, and let $\eta_A : A \to \mathbb{Q}$ be the natural map $(a_1, a_2, ..., a_n) \to a_1 + a_2 + \cdots + a_n$. Then $G(A_1, A_2, ..., A_n) = \operatorname{Ker} \eta_A$ is clearly a Butler group of corank 1, and moreover, as Richman observed in [R], each Butler group of corank 1 is isomorphic to the direct sum of a finite rank completely decomposable group and some appropriate $G(A_1, A_2, ..., A_n)$. Thus, we may restrict ourselves to corank 1 Butler groups of the special form $G(A_1, A_2, ..., A_n)$. The doubly incomparable groups referred to in the introduction are, in fact, of this form.

With the notation of the preceding paragraph in effect, let $G = G(A_1, A_2, ..., A_n)$ and with an arbitrary nonzero element $c \in \bigcap_{i \leq n} A_i$ associate its height vector $(|a_1|, |a_2|, ..., |a_n|)$, where $c = a_i \in A_i$ and $|a_i|$ is the height of a_i as computed in A_i. It will be convenient to denote the height vector of c by $|\mathbf{c}|$. Notice that each of the heights $|a_1|, |a_2|, ..., |a_n|$ is contained in the height spectrum of $c + G = a_1 + G = a_2 + G = \cdots = a_n + G$. Even though $\|c + G\|$ will generally contain other heights, the height vector $|\mathbf{c}| = (|a_1|, |a_2|, ..., |a_n|)$ will, as is demonstrated in Theorem 2.4 below, effectively determine $\|c + G\|$. By a height vector (not necessarily associated with an element c) we simply mean an n-tuple of heights. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, ..., \beta_n)$ denote general height vectors. We say that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are equivalent if there exist positive integers m and k such that $m\boldsymbol{\alpha} = k\boldsymbol{\beta}$, where $m\boldsymbol{\alpha} = (m\alpha_1, m\alpha_2, ..., m\alpha_n)$ and the integer m operates on a height α_i in the obvious manner. Not surprisingly, we call the equivalence class represented by $\boldsymbol{\alpha}$ the type of the height vector $\boldsymbol{\alpha}$, or the type vector associated with $\boldsymbol{\alpha}$.

Consider the exact sequence

$$0 \to G(A_1, A_2, ..., A_n) \to \bigoplus_{i \leq n} A_i \to \sum_{i \leq n} A_i \to 0 \tag{E}$$

Clearly any two nonzero elements c and c' of $\bigcap_{i \leq n} A_i$ have equivalent height vectors. Therefore, the exact sequence (E) has a unique and well defined type vector associated with it. Equivalently, if G is a pure subgroup of corank 1 in a completely decomposable group $A = \bigoplus_{i \leq n} A_i$ where $A_i \subseteq \mathbb{Q}$ and no A_i is contained in G then there is a unique type vector associated with the pair (G, A). Conversely, if $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ represents an arbitrary type vector, then there exists an n-tuple $(A_1, A_2, ..., A_n)$ of subgroups of $\mathbb{Q}$ such that the corresponding exact sequence (E) has that given type vector, for we can take A_i to be the subgroup of $\mathbb{Q}$ containing 1 and for which the height of 1 is precisely α_i.

Not only does the exact sequence (E) determine uniquely a type vector, but the following theorem (already implicit in [R, Theorem 2.1]) shows that a type vector has an essentially unique associated exact sequence (E).

THEOREM 2.4. *Let $(A_1, A_2, ..., A_n)$ and $(B_1, B_2, ..., B_n)$ be two n-tuples of nonzero subgroups of $\mathbb{Q}$ and consider the exact sequences*

$$0 \longrightarrow G(A_1, A_2, ..., A_n) \longrightarrow \bigoplus_{i \leq n} A_i \xrightarrow{\eta_A} \sum_{i \leq n} A_i \longrightarrow 0 \tag{E}$$

and

$$0 \longrightarrow G(B_1, B_2, ..., B_n) \longrightarrow \bigoplus_{i \leqslant n} B_i \xrightarrow{\eta_B} \sum_{i \leqslant n} B_i \longrightarrow 0 \qquad (\mathrm{E}')$$

If (E) *and* (E') *have the same type vector, then they are equivalent exact sequences through isomorphisms $A_i \to B_i$. In particular, $G(A_1, A_2, ..., A_n)$ and $G(B_1, B_2, ..., B_n)$ are isomorphic.*

Proof. Suppose that (E) and (E′) have the same type vector. Then there exist nonzero elements $c \in \bigcap_{i \leqslant n} A_i$ and $c' \in \bigcap_{i \leqslant n} B_i$ with the same height vector $|\mathbf{c}| = |\mathbf{c}'|$. This means that the height of c in A_i is equal to the height of c' in B_i for each $i \leqslant n$. Therefore, there is an isomorphism $\pi_i \colon A_i \to B_i$ that maps c to c' for all i. If we set $\pi = \bigoplus_{i \leqslant n} \pi_i$, it remains only to verify that π maps $G = G(A_1, A_2, ..., A_n)$ onto $G' = G(B_1, B_2, ..., B_n)$ in order to complete the proof that (E) and (E′) are equivalent. Actually, it suffices to prove that $\pi(G) \subseteq G'$, due to symmetry (or to the fact that $\bigoplus_{i \leqslant n} A_i$ and $\bigoplus_{i \leqslant n} B_i$ have finite rank). Moreover, since G and G' are pure, it is enough to demonstrate that $\pi(mg) \in G'$ for some positive integer m whenever $g \in G$. Thus let $g = (a_1, a_2, ..., a_n)$ denote an arbitrary element of G and choose a positive integer m such that $ma_j \in \bigcap_{i \leqslant n} A_i$ for each $j \leqslant n$. For $j \leqslant n-1$, let g_j denote that element of G having ma_j as its jth component, $-ma_j$ as its nth component, and all other components zero. Observe, since $\sum_{i \leqslant n} a_i = 0$, that $mg = \sum_{j \leqslant n-1} g_j$. Furthermore, increasing m if necessary, we may select nonzero integers k_j such that $k_j c = ma_j$ for all $j \leqslant n-1$. Hence $\pi(g_j) = (..., \pi_j(ma_j), ..., -\pi_n(ma_j)) = (..., k_j c', ..., -k_j c')$ is an element of G' for each $j \leqslant n-1$. Therefore $\pi(mg) \in G'$, and the theorem is proved.

Since the isomorphism π constructed in the proof of Theorem 2.4 maps $G(A_1 A_2, ..., A_n)$ to $G(B_1, B_2, ..., B_n)$, it induces a map between the corresponding quotients that preserves height spectrums. Under certain circumstances, such a map between quotients determines the corresponding type vector. We call an n-tuple $(A_1, A_2, ..., A_n)$ of subgroups of $\mathbb{Q}$ incomparable if type(A_1), type(A_2), ..., type(A_n) are pairwise incomparable; and, following Richman [R], we call an n-tuple $(A_1, A_2, ..., A_n)$ doubly incomparable if type($A_i \cap A_j$) and type($A_k \cap A_l$) are incomparable whenever the pairs (i, j) and (k, l) are distinct.

THEOREM 2.5. *Suppose $(A_1, A_2, ..., A_n)$ and $(B_1, B_2, ..., B_m)$ are, respectively, an incomparable n-tuple and m-tuple of subgroups of $\mathbb{Q}$, and let $G = G(A_1\ A_2, ..., A_n)$ and $G' = G(B_1, B_2, ..., B_m)$. If there is an isomorphism $\phi \colon A/G \to B/G'$ that preserves height spectrums, then $n = m$ and there is a permutation of the i's such that the associated exact sequences* (E) *and* (E′) *have the same type vector. In particular, $G \cong G'$.*

Proof. First observe, since $G \cap A_i = 0$ and $G' \cap B_j = 0$, that the hypotheses on the A_i's and B_j's insure that $G \cap A(\sigma) \subseteq A(\sigma^*)$ and $G' \cap B(\sigma) \subseteq B(\sigma^*)$ for all types σ. Then, Corollary 1.6 implies that there exists an isomorphism $\pi: A \to B$ such that $\pi(G) = G'$. Therefore $G \cong G'$ and $n = \operatorname{rank}(A) = \operatorname{rank}(B) = m$. Moreover, the hypotheses imply that the A_i's and B_j's are fully invariant subgroups of A and B, respectively. Thus, after a permutation of the i's, we may assume that $\pi(A_i) = B_i$ for all $i \leqslant n$. Finally, the conclusion about the type vectors follows from our earlier observation about the uniqueness of the type vector associated with a given exact sequence (E).

It is, of course, possible to convert a type vector to a purely numerical quantity. For example, one approach is the following. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ represent a given type vector and, for each prime p, let

$$k_p = \min\{\alpha_{ip} : \alpha_{ip} \neq \infty\}.$$

Then define a matrix E_A, with columns indexed by $\mathbf{P}$ and rows indexed by $i = 1, 2, ..., n$, by taking

$$E_A(i, p) = \begin{cases} \infty & \text{if} \quad \alpha_{i_p} = \infty \\ \alpha_{i_p} - k_p & \text{if} \quad \alpha_{i_p} \neq \infty. \end{cases}$$

The matrix E_A is closely related to the invariant introduced, somewhat differently, in [R] to classify $G(A_1, A_2, ..., A_n)$ for the case where $(A_1, A_2, ..., A_n)$ is a doubly incomparable n-tuple of subgroups of $\mathbb{Q}$.

Though the type vector (or some variant of it such as that discussed in the preceding paragraph) determines the exact sequence (E) up to equivalence and hence determines $G = G(A_1, A_2, ..., A_n)$ up to isomorphism, it fails to be defined intrinsically in terms of the structure of G and hence may fail to be an invariant of G. It can easily happen that $G(A_1, A_2, ..., A_n) \cong G(B_1, B_2, ..., B_n)$ where the exact sequences (E) and (E′) of Theorem 3.4 have different type vectors. Indeed such an example (see 1.4 in [R]) already appears in the literature where $n = 4$ and the 4-tuples are incomparable. This is less likely, but can still happen, if we insist that (E) and (E′) are trimmed in the sense of [R]: the n-tuple $(A_1, A_2, ..., A_n)$ is trimmed if the projection of $G(A_1, A_2, ..., A_n)$ into A_i is onto for each $i \leqslant n$. An equivalent formulation is that $A_i = A_i \cap \sum_{j \neq i} A_j$ hold for all $i \leqslant n$. Clearly if A_i' is the projection of $G(A_1, A_2, ..., A_n)$ into A_i, then $G(A_1', A_2', ..., A_n') = G(A_1, A_2, ..., A_n)$ and $(A_1', A_2', ..., A_n')$ is a trimmed n-tuple of subgroups of $\mathbb{Q}$. Therefore, there is no loss of generality in assuming that (E) is trimmed. An obvious question that we leave unresolved is the following: If (E) and (E′) are trimmed in Theorem 2.4, what are necessary and sufficient conditions on their type vectors in order that $G(A_1, A_2, \ldots, A_n)$ and $G(B_1, B_2, \ldots, B_n)$ be isomorphic? Instead we shall consider the following equally natural question. When is the type vector an invariant of $G = G(A_1, A_2, \ldots, A_n)$?

Assuming that $(A_1, A_2, \ldots, A_n)$ is a trimmed n-tuple of subgroups of $\mathbb{Q}$, we have a canonical isomorphism $A_i \cong G/G^{(i)}$, for each i, where $G^{(i)} = G \cap \bigoplus_{j \neq i} A_j$. Thus, there is an induced commutative diagram

$$\begin{tikzcd} & \bigoplus_{i=1}^{n} G/G^{(i)} \arrow[dd] \\ G \arrow[ur] \arrow[dr] & \\ & \bigoplus_{i=1}^{n} A_i \end{tikzcd}$$

where all maps are natural. Therefore, in order for the type vector to be an invariant of G, the quotients $G/G^{(i)}$ should be invariants of G. This suggests that the subgroups $G^{(i)}$ should be fully invariant. But since $G^{(i)}$ is the kernel of a homomorphism from G onto a rank 1 group of type $\mu_i = \text{type}(A_i)$, there is an obvious candidate. Recall that for an arbitrary type σ, the σ-radical $G[\sigma]$ is defined as $\bigcap \{\text{Ker}\,\phi : \phi \in \text{Hom}(G, X_\sigma)\}$ where X_σ is fixed rank 1 group of type σ. Hence, in general, $G[\mu_i] \subseteq G^{(i)}$; and we are led almost inevitably to the consideration of that special class of groups $G = G(A_1, A_2, \ldots, A_n)$ that possess the property that, for each i, $G \cap \bigoplus_{j \neq i} A_j = G[\mu_i]$ where $\mu_i = \text{type}(A_i)$. In fact, as we shall see below, Richman's doubly incomparable groups are contained in this class.

The computation of $G[\sigma]$ for groups of the form $G = G(A_1, A_2, \ldots, A_n)$ has a conspicuously simple description recorded in the next lemma (which also appears as Corollary 1.11(c) in [AV2]).

Lemma 2.6. *Suppose* $G = G(A_1, A_2, \ldots, A_n)$ *where* $\mu_i = \text{type}(A_i)$ *for each* $i = 1, 2, \ldots, n$, *and let* $G_{ij} = G \cap (A_i \oplus A_j)$ *whenever* $1 \leqslant i < j \leqslant n$. *Then for any type* σ,

$$G[\sigma] = \langle G_{ij} : \mu_i \wedge \mu_j \nleqslant \sigma \rangle_*.$$

Proof. Notice, from the definition of $G = G(A_1, A_2, \ldots, A_n)$, that $G_{ij} \cong A_i \cap A_j$ and therefore that G_{ij} is a rank 1 pure subgroup of G having type $\mu_i \wedge \mu_j$. Thus when $\mu_i \wedge \mu_j \nleqslant \sigma$, it is clear that G_{ij} will be contained in the kernel of every homomorphism from G into a rank 1 group of type σ. Hence, inclusion in one direction is trivial.

To establish the reverse inclusion, suppose $g \in G(\tau)$ where $\tau \nleq \sigma$. Since the canonical map $\bigoplus G_{ij} \to G$ is a balanced epimorphism [AV1, Theorem 1.2], we have a representation $g = \sum g_{ij}$ where $g_{ij} \in G_{ij}$ and, when $g_{ij} \neq 0$, $\tau \leqslant \text{type}(g) \leqslant \text{type}(g_{ij}) = \mu_i \wedge \mu_j$. It follows that $g \in \langle G_{ij} : \mu_i \wedge \mu_j \nleq \sigma \rangle$. Then, the fact that $G[\sigma] = \langle G(\tau) : \tau \nleq \sigma \rangle_*$ [AV1, Proposition 1.9] yields the desired conclusion.

From 2.6, it is immediate that $G = G(A_1, A_2, ..., A_n)$ will enjoy the special property that $G^{(k)} = G[\mu_k]$, for each $k = 1, 2, ..., n$, provided $G \cap \bigoplus_{j \neq k} A_j$ is the pure closure of those G_{ij}'s such that $\mu_i \wedge \mu_j \nleq \mu_k$. As it turns out, there is a neat geometrical description of the groups with this property. It is simple to check that Richman's doubly incomparable groups and also the groups in his Example 1.4 [R] satisfy the following striking condition: For each $k = 1, 2, ..., n$, there is a fixed $i = i(k) \neq k$ such that $\mu_i \wedge \mu_j \nleq \mu_k$ whenever $j \neq k$. Viewing the foregoing condition geometrically, we see that the pairs ij can be thought of as spokes on a wheel with $i = i(k)$ as hub. This observation serves as the inspiration for the following definition.

Definition 2.7. Let $\Omega_n = \{1, 2, ..., n\}$, and suppose $G \cong G(A_1, A_2, ..., A_n)$ where $(A_1, A_2, ..., A_n)$ is a trimmed n-tuple of subgroups of $\mathbb{Q}$ and $\mu_i = \text{type}(A_i)$ for each $i \in \Omega_n$. We call G a *Richman–Butler group* provided, for each $k \in \Omega_n$, there is a connected graph P_k with vertex set $I_k = \Omega_n \setminus \{k\}$ such that $\mu_i \wedge \mu_j \nleq \mu_k$ whenever ij is an edge of P_k.

Theorem 2.8. *If G is a Richman–Butler group of rank $n-1$, then there exists a set of incomparable types $\{\mu_1, \mu_2, ..., \mu_n\}$ such that, for each i, $G/G[\mu_i]$ is a* rank 1 *group of type μ_i and the natural map $G \to \bigoplus_{i=1}^n G/G[\mu_i]$ is a pure imbedding. Furthermore, the type vector associated with this imbedding is an invariant that determines G up to isomorphism.*

Proof. Let $G = G(A_1, A_2, ..., A_n)$ be a representation satisfying the conditions of Definition 2.7. It is not difficult to see that

$$G^{(k)} = \langle G_{ij} : ij \text{ is an edge of } P_k \rangle_*$$

(see, for example, Lemma 1.4 in [AV2]). Then from Lemma 2.6 and the fact that $G/G^{(k)} \cong A_k$, it follows that $G^{(k)} = G[\mu_k]$. Notice that the μ_i's are incomparable since if $\mu_i \leqslant \mu_k$ for $i \neq k$, then we would have $\mu_i \wedge \mu_j \nleq \mu_k$ contradicted for ij an edge of P_k. The remaining assertions follow, under appropriate identifications, from Theorem 2.4.

Unfortunately, Theorem 2.8 is less satisfactory than it may appear at first blush. The problem is that a Richman-Butler group may have many different representations leading to different type vectors. We shall, however, surmount these difficulties in the next section where we describe the extended canonical imbedding for Richman–Butler groups. We will then be able to assign to each Richman–Butler group a unique *extended type vector* which, together with certain other cardinal invariants, will completely determine a Richman–Butler group up to isomorphism.

3. The Uniqueness Theorem for Richman–Butler Groups

We begin by laying down certain notational conventions that will be in effect throughout this section. First, for any positive integer n, $\Omega_n = \{1, ..., n\}$. Whenever we employ the notation $G(A_1, ..., A_n)$, it is tacitly assumed that $(A_1, ..., A_n)$ is a given n-tuple of nonzero subgroups of $\mathbb{Q}$. On the other hand, in those few instances where we require $(A_1, ..., A_n)$ to be trimmed, this assumption will be explicitly stated. As in Lemma 2.6 above, when $G = G(A_1, ..., A_n)$ and $1 \leqslant i < j \leqslant n$, G_{ij} will denote the rank 1 pure subgroup $G \cap (A_i \oplus A_j)$. It is not until Theorem 3.7 that we require the groups under consideration to be Richman–Butler groups and hence, up to that point, all results apply to arbitrary groups of the form $G(A_1, ..., A_n)$.

As preliminary orientation for the reader, we now give a rough sketch of the fundamental ideas underlying this section. For any group G of the form $G(A_1, ..., A_n)$, we define a canonical map from G to a finite rank completely decomposable group A_G, where A_G denotes the direct sum of all those quotients $G/G[\mu]$ that happen to be rank 1. When G is a Richman–Butler group, this map will be a pure imbedding and the extended type vector, as extracted from A_G, will prove to be an invariant of G. The extended type vector and the ranks of certain fully invariant subgroups will then constitute a complete set of isomorphism invariants for the Richman–Butler group G. Furthermore, any other Richman–Butler group G' with the same invariants will be imbedded in an $A_{G'} \cong A_G$ and a delicate study of these imbeddings will actually lead to the existence of an isomorphism from A_G to $A_{G'}$ that maps G onto G'.

We shall first require, for an arbitrary $G = G(A_1, ..., A_n)$ and type σ, what might be termed a geometrical analysis of the quotient $G/G[\sigma]$. Although the next theorem can be derived from Theorem 1.10 in [AV2] and Corollary 3.3 below can be viewed as a special case of Theorem 2.2 in the same paper, it should be emphasized that we do not use the concept of a *quasi-representing graph*, a notion which in the present context seems as much burden as help.

Theorem 3.1. *Let $G = G(A_1, ..., A_n)$ and $\mu_i = \text{type}(A_i)$ for each $i \in \Omega_n$. Then to each type σ there corresponds a family of connected graphs*

$P_1, \ldots, P_m$ with vertex sets $I_1, \ldots, I_m$, respectively, such that the following conditions are satisfied:

(1) *The sets $I_1, \ldots, I_m$ partition Ω_n.*

(2) *If ij is an edge of some P_k, then $\mu_i \wedge \mu_j \nleq \sigma$.*

(3) *$G[\sigma] = G_1 \oplus \cdots \oplus G_m$ where $G_k = G \cap \bigoplus_{i \in I_k} A_i$ for each $k \in \Omega_m$.*

(4) *The subgroup $G[\sigma]$ is the kernel of a homomorphism ρ from G onto $\bar{G} = G(B_1, \ldots, B_m)$ where, for each $k \in \Omega_m$, $B_k = (\Sigma_{i \in I_k} A_i) \cap (\sum_{j \notin I_k} A_j)$.*

Proof. Define an equivalence relation $\sim$ on Ω_n such that $i \sim j$ for distinct $i, j \in \Omega_n$ if and only if there is a finite sequence $i = i_1, \ldots, i_s = j$ in Ω_n such that $\mu_{i_t} \wedge \mu_{i_{t+1}} \nleq \sigma$ for $t = 1, \ldots, s-1$. Take $I_1, \ldots, I_m$ to be the equivalence classes modulo $\sim$ and note that the existence of the P_k's is obvious. Now define the map ρ as follows: If $x = (a_1, \ldots, a_n) \in G$, then $\rho(x) = (b_1, \ldots, b_m)$ where $b_k = \sum_{i \in I_k} a_i$ for each $k \in \Omega_m$. It is clear that $\operatorname{Ker} \rho = G_1 \oplus \cdots \oplus G_m$, and it remains only to show that $\operatorname{Ker} \rho = G[\sigma]$. This, however, follows from Lemma 2.6 and the observation that $G_k = \langle G_{ij} : ij \text{ is an edge of } P_k \rangle_*$ for each $k \in \Omega_m$.

Remark. The m-tuple $(B_1, \ldots, B_m)$ constructed in 3.1 is trimmed, even if the original n-tuple $(A_1, \ldots, A_n)$ is not. To see this, first note that, because of the distributivity of the lattice of subgroups of $\mathbb{Q}$, B_k is the group union of all the subgroups $A_i \cap A_j$ where $i \in I_k$ and $j \notin I_k$. For a given such pair (i, j), choose l such that $j \in I_l$ and observe, by definition, that $A_j \cap A_i \subseteq B_l$. Therefore, for each $k \in \Omega_m$, $B_k \subseteq \sum_{l \neq k} B_l$ and consequently $(B_1, \ldots, B_m)$ is a trimmed m-tuple of subgroups of $\mathbb{Q}$.

Definition 3.2. Recall that a type μ is said to be a cotype of the torsion-free group G if G has a rank 1 homomorphic image of type μ. For any torsion-free group G, $\Delta(G)$ denotes the set of all those cotypes μ of G such that rank $(G/G[\mu]) = 1$.

Corollary 3.3. *Let $G = G(A_1, \ldots, A_n)$ and $\mu_i = \text{type}(A_i)$ for all $i \in \Omega_n$. Then $\mu \in \Delta(G)$ if and only if there are connected graphs P_μ and Q_μ with vertex sets $I_\mu = \{i_1, \ldots, i_s\}$ and $J_\mu = \{j_1, \ldots, j_t\}$, respectively, such that the following conditions are satisfied:*

(a) *$I_\mu \cap J_\mu = \varnothing$ and $I_\mu \cup J_\mu = \Omega_n$.*

(b) *If ij is an edge of either P_μ or Q_μ, then $\mu_i \wedge \mu_j \nleq \mu$.*

(c) *$G[\mu] = G(A_{i_1}, \ldots, A_{i_s}) \oplus G(A_{j_1}, \ldots, A_{j_t})$.*

(d) *$\mu = (\bigvee_{i \in I_\mu} \mu_i) \wedge (\bigvee_{j \in J_\mu} \mu_j)$.*

Proof. First suppose $\mu \in \Delta(G)$ and take $\sigma = \mu$ in Theorem 3.1. Since $n-2 = \operatorname{rank}(G[\mu]) = \operatorname{rank}(G[\sigma]) = \sum_{k=1}^{m} (|I_k| - 1)$ and $n = \sum_{k=1}^{m} |I_k|$, we must have $n=2$. Take $P_\mu = P_1$, $Q_\mu = P_2$, $I_\mu = I_1$, $J_\mu = I_2$ and observe that (a), (b), (c) are satisfied. Finally note, since $m=2$, that $B_1 = B_2$ is a group of type $(\bigvee_{i \in I_\mu} \mu_i) \wedge (\bigvee_{j \in J_\mu} \mu_j)$ and $G/G[\mu] \cong \bar{G} = G(B_1, B_2) \cong B_1$ must have type μ. Thus (d) follows.

Conversely, assume that the conditions are satisfied and take $\sigma = s$ in Theorem 3.1. Without loss of generality, $I_1 \cap I_\mu \neq \varnothing$ and the definition of $\sim$ implies that $I_\mu \subseteq I_1$. If this inclusion were proper, then it is easy to see that there would be an edge ij of P_1 such that $i \in I_\mu$ and $j \in J_\mu$. But then by (d), we have the contradiction that $\mu_i \wedge \mu_j \leqslant \mu = \sigma$ (see 3.1(2)). Thus $I_1 = I_\mu$ and consequently $I_2 \subseteq J_\mu$ by (a). It follows that $I_2 = J_\mu$ from the definition of $\sim$. Thus once again we are forced to conclude that $m=2$ and that $G/G[\mu] \cong \bar{G} = G(B_1, B_2)$ is a rank 1 group of type $((\bigvee_{i \in I_\mu} \mu_i) \wedge (\bigvee_{j \in J_\mu} \mu_j)$. Finally, (d) is required again to confirm that μ is indeed a cotype of G.

COROLLARY 3.4. *Let $G = G(A_1, ..., A_n)$, $\bar{G} = G(B_1, ..., B_m)$ and $\rho\colon G \to \bar{G}$ be as in the statement of Theorem* 3.1. *Take $\bar{\mu}_k = \operatorname{type}(B_k)$ for each $k \in \Omega_m$. Then $\Delta(\bar{G}) = \{\mu \in \Delta(G) : \mu \leqslant \sigma\}$. Furthermore if $\mu \in \Delta(\bar{G})$ and the sets I_μ and J_μ are as in the statement of Corollary* 3.3, *then there are connected graphs $\bar{P}_\mu$ and $\bar{Q}_\mu$ with vertex sets $\bar{I}_\mu = \{k : I_k \cap I_\mu \neq \varnothing\}$ and $\bar{J}_\mu = \{k : I_k \cap J_\mu \neq \varnothing\}$, respectively, such that the following conditions are satisfied*:

($\bar{\text{a}}$) $\bar{I}_\mu \cap \bar{J}_\mu = \varnothing$ *and* $\bar{I}_\mu \cup \bar{J}_\mu = \Omega_m$.

($\bar{\text{b}}$) *If kl is an edge of either $\bar{P}_\mu$ or $\bar{Q}_\mu$, then $\bar{\mu}_k \wedge \bar{\mu}_l \nleqslant \mu$.*

($\bar{\text{c}}$) $\bar{G}[\mu] = (\bar{G} \cap \bigoplus_{k \in \bar{I}_\mu} B_k) \oplus (\bar{G} \cap \bigoplus_{k \in \bar{J}_\mu} B_k)$.

Proof. Observe, by (c) and (e) of Lemma 1.3 in [AV1], that if μ is a cotype of $\bar{G}$, then $\mu \leqslant \sigma$. But then $(G/G[\sigma])[\mu] = G[\mu]/G[\sigma]$ and $\bar{G}/\bar{G}[\mu] \cong G/G[\mu]$. Since $\bar{G}$ is a homomorphic image of G, the above description of $\Delta(\bar{G})$ follows.

By Corollary 3.3 applied to $\bar{G}$, we at least know that there are connected graphs $\bar{P}_\mu$ and $\bar{Q}_\mu$ with certain vertex sets $\bar{I}_\mu$ and $\bar{J}_\mu$, respectively, such that conditions ($\bar{\text{a}}$), ($\bar{\text{b}}$) and ($\bar{\text{c}}$) are satisfied. The real significance of the present result is that $\bar{I}_\mu$ and $\bar{J}_\mu$ have the above description in terms of I_μ and J_μ. First observe that if $I_k \cap I_\mu \neq \varnothing$, then $I_k \subseteq I_\mu$ because otherwise P_k would have an edge ij with $i \in I_\mu$, $j \in J_\mu$ and this would imply $\mu_i \wedge \mu_j \leqslant \mu \leqslant \sigma$. Since the same holds for J_μ, we see that

$$I_\mu = \bigcup \{I_k : I_k \cap I_\mu \neq \varnothing\} \qquad \text{and} \qquad J_\mu = \bigcup \{I_k : I_k \cap J_\mu \neq \varnothing\}.$$

Now take $I=\{k : I_k \cap I_\mu \neq \varnothing\}$ and $J=\{k : I_k \cap J_\mu \neq \varnothing\}$. Then $I\cap J=\varnothing$ and $I\cup J=\Omega_m$. Without loss of generality, we may suppose that $I\cap \bar{I}_\mu \neq \varnothing$. We claim that it follows that $\bar{I}_\mu \subseteq I$. If not, then the connected graph $\bar{P}_\mu$ would have an edge kl where $k\in I$ and $l\in J$. Then, since we would have $I_k \subseteq I_\mu$ and $I_l \subseteq J_\mu$, $\bar{\mu}_k = (\bigvee_{i\in I_k} \mu_i) \wedge (\bigvee_{j\notin I_k} \mu_j) \leqslant \bigvee_{i\in I_k} \mu_i \leqslant \bigvee_{i\in I_\mu} \mu_i$ and similarly $\bar{\mu}_l \leqslant \bigvee_{j\in J_\mu} \mu_j$. But then by (d) of Corollary 3.3, $\bar{\mu}_k \wedge \bar{\mu}_l \leqslant \mu$ which contradicts (b) above. Notice now that $\bar{I}_\mu \subseteq I$ implies $\bar{J}_\mu \supseteq J$. In particular $J\cap \bar{J}_\mu \neq \varnothing$ and, by an argument similar to that given immediately above, it follows that $\bar{J}_\mu \subseteq J$. Thus $J=\bar{J}_\mu$ and, consequently, also $I=\bar{I}_\mu$. This completes the proof.

DEFINITION 3.5. Suppose $\mu\in \Delta(G)$ where $G=G(A_1, ..., A_n)$ and let $A_\mu = (\sum_{i\in I_\mu} A_i) \cap (\sum_{j\in J_\mu} A_j)$ where I_μ and J_μ are as in Corollary 3.3. Now take $A_G = \bigoplus_{\mu\in \Delta(G)} A_\mu$ and, for each μ, let π_μ be the canonical projection of A_G onto A_μ. Define $\theta_G : G\to A_G$ to be that unique homomorphism such that $\pi_\mu \theta_G(x) = \sum_{i\in I_\mu} a_i$ whenever $x=(a_1, ..., a_n)\in G$.

Notice that the inherent ambiguity between I_μ and J_μ introduces an ambiguity in the definition of θ_G. But for our purposes, this proves not to be a difficulty since a switch between I_μ and J_μ results only in a change of sign for $\pi_\mu\theta_G(x)$. Observe also that $\pi_\mu\theta_G$ is a homomorphism of G onto A_μ with $G[\mu]$ as kernel.

COROLLARY 3.6. *Let* $G=G(A_1, ..., A_n)$, $\bar{G}=G(B_1, ..., B_m)$ *and* $\rho: G\to \bar{G}$ *be as in the statement of Theorem* 3.1, *then* $A_{\bar{G}} = \bigoplus \{A_\mu : \mu \leqslant \sigma\}$ *and we have a commutative diagram*

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \bar{G} \\ \Big\downarrow{\scriptstyle \theta_G} & & \Big\downarrow{\scriptstyle \theta_{\bar{G}}} \\ A_G & \xrightarrow{\pi} & A_{\bar{G}} \end{array}$$

where π *is the canonical projection of* A_G *onto its summand* $A_{\bar{G}}$.

Proof. Since $A_{\bar{G}} = \bigoplus_{\mu\in \Delta(\bar{G})} B_\mu$ where $B_\mu = (\sum_{k\in I_\mu} B_k) \cap (\sum_{l\in J_\mu} B_l)$, Corollary 3.4 yields $B_\mu = X\cap Y$ where $X=\sum_{I_k\cap I_\mu\neq\varnothing} (\sum_{i\in I_k} A_i \cap \sum_{j\notin I_k} A_j)$ and $Y=\sum_{I_k\cap J_\mu\neq\varnothing} (\sum_{j\in I_k} A_j \cap \sum_{i\notin I_k} A_i)$. Obviously $B_\mu \subseteq A_\mu$ since $X\subseteq \sum_{i\in I_\mu} A_i$ and $Y\subseteq \sum_{j\in J_\mu} A_j$. The reverse inclusion $A_\mu \subseteq B_\mu$ follows from the fact that $A_\mu = \sum\{A_i\cap A_j : (i, j)\in I_\mu\times J_\mu\}$, because of the distributivity of the subgroup lattice of $\mathbb{Q}$, and the immediate fact that $A_i\cap A_j \subseteq X\cap Y$ for each $(i, j)\in I_\mu \times J_\mu$. Thus, we have established the initial claim that $A_{\bar{G}} = \bigoplus \{A_\mu : \mu\leqslant\sigma\}$. From the definition of ρ, as given in the proof of

Theorem 3.1, the commutativity of the diagram is quickly checked by the following simple calculation:

$$\pi_\mu \theta_G(x) = \sum_{i \in I_\mu} a_i = \sum_{I_k \cap I_\mu \neq \varnothing} \left(\sum_{i \in I_k} a_i \right) = \sum_{k \in \bar{I}_\mu} b_k = \pi_\mu \theta_{\bar{G}} \rho(x)$$

where $x = (a_1, ..., a_n) \in G$.

As our next theorem shows, if G is a Richman–Butler group with $|\Delta(G)| \geqslant \operatorname{rank}(G) + 2$, then G has numerous proper homomorphic images that are also Richman–Butler groups.

THEOREM 3.7. *Let $\mu \in \Delta(G)$ where $G \cong G(A_1, ..., A_n)$ is a Richman–Butler group. Suppose that I_μ and J_μ are as in Corollary 3.3 and that both I_μ and J_μ contain at least two elements. Then if $\sigma = \bigvee_{i \in I_\mu} \mu_i$, $G/G[\sigma]$ is a Richman–Butler group isomorphic to $\bar{G} = G(A_{i_1}, ..., A_{i_s}, A_\mu)$.*

Proof. Let $\sigma' = \bigvee_{j \in J_\mu} \mu_j$ so that $\sigma \wedge \sigma' = \mu$. If $\sim$ is the equivalence relation as defined in the proof of Theorem 3.1, then we claim that the equivalence classes modulo $\sim$ are just $I_k = \{i_k\}$ for $k = 1, ..., s$ and $I_{s+1} = J_\mu$. To see this, first observe that if ij is an edge of the graph Q_μ, then $\mu_i \wedge \mu_j \nleqslant \mu = \sigma \wedge \sigma'$; but since $\mu_i \wedge \mu_j \leqslant \sigma'$, we actually have $\mu_i \wedge \mu_j \nleqslant \sigma$. On the other hand, if $i \in I_\mu$ and $j \in \Omega_n \setminus \{i\}$, then we maintain that $\mu_i \wedge \mu_i \leqslant \sigma$. Indeed if $j \in I_\mu$, this is clear from the definition of σ; while if $j \in J_\mu$, then $\mu_i \wedge \mu_j \leqslant \mu \leqslant \sigma$. Therefore Theorem 3.1 applies to yield a homomorphism ρ of G onto $\bar{G} = G(A_{i_1}, ..., A_{i_s}, A_\mu)$ where $A_\mu = (\sum_{i \in I_\mu} A_i) \cap (\sum_{j \in J_\mu} A_j)$ and $\operatorname{Ker} \rho = G[\sigma] = G(A_{j_1}, ..., A_{j_t})$. Notice that $\operatorname{rank}(G/G[\sigma]) = s$ where $2 \leqslant s \leqslant n-2$.

We shall show that $G/G[\sigma]$ is a Richman–Butler group by demonstrating that $\bar{G} = G(A_{i_1}, ..., A_{i_s}, A_\mu)$ satisfies the requisite conditions formulated in Definition 2.7. As we may assume that the n-tuple $(A_1, .., A_n)$ satisfies those conditions, for each $k \in I_\mu$ there is a connected graph P_k with vertex set $\Omega_n \setminus \{k\}$ such that $\mu_i \wedge \mu_j \nleqslant \mu_k$ whenever ij is an edge of P_k. From P_k we construct a connected graph $\bar{P}_k$ with vertex set $(I_\mu \setminus \{k\}) \cup \{\mu\}$ as follows: If ij is an edge of P_k with both i and j in I_μ, then ij is also an edge of $\bar{P}_k$; whereas if ij is an edge of P_k with $i \in I_\mu$ and $j \in J_\mu$, then $i\mu$ is an edge of $\bar{P}_k$. Thus if ij is an edge of both $\bar{P}_k$ and P_k, then $\mu_i \wedge \mu_j \nleqslant \mu_k$. If on the other hand, $i\mu$ is an edge of $\bar{P}_n$ coming from an edge ij of P_k with $j \in J_\mu$, then $\mu_i \wedge \mu_j \leqslant \mu$ and consequently $\mu_i \wedge \mu \nleqslant \mu_k$, as desired, because $\mu_i \wedge \mu_j \nleqslant \mu_k$. Finally, to complete the proof that $\bar{G}$ is a Richman–Butler group, we take $\bar{P}_\mu = P_\mu$ where the latter is as in Corollary 3.3.

From a quite different perspective, Arnold and Vinsonhaler [AV3] have also investigated the class of groups that we have referred to as Richman–Butler groups. Recall that two finite rank torsion-free groups G and G' are *quasi-isomorphic* if there exist monomorphisms $\phi: G \to G'$ and $\phi': G' \to G$, and that a torsion-free group is said to be *strongly indecomposable* if it is not quasi-isomorphic to any group with a nontrivial direct decomposition. The fundamental significance of this latter class of groups is the observation, due to Jónsson [J], that every finite rank torsion-free group is quasi-isomorphic to direct sum of strongly indecomposable groups and that the Krull–Schmidt theorem holds, up to quasi-isomorphism, for such decompositions.

THEOREM 3.8 [AV3]. *Suppose $G = G(A_1, ..., A_n)$ where $(A_1, ..., A_n)$ is a trimmed n-tuple of subgroups of $\mathbb{Q}$ and let $\mu_i = \text{type}(A_i)$ for each $i \in \Omega_n$. Then G is strongly indecomposable if and only if $\text{rank}(G/G[\mu_i]) = 1$ for all $i \in \Omega_n$.*

The proof, given in [AV3], that $G = G(A_1, ..., A_n)$ being strongly indecomposable implies that each $G/G[\mu_i]$ has rank 1 relies on an earlier study of quasi-decompositions of groups of the special form $G(A_1, ..., A_n)$. A more direct *ad hoc* proof is, no doubt, possible. For the converse, Arnold and Vinsonhaler [AV3] provide a computation showing, under the hypothesis that each $G/G[\mu_i]$ is rank 1, that the endomorphism ring of $G = G(A_1, ..., A_n)$ is a subring of $\mathbb{Q}$, which is well known to imply that G is strongly indecomposable.

THEOREM 3.9. *Suppose $G = G(A_1, ..., A_n)$ where $(A_1, ..., A_n)$ is a trimmed n-tuple of subgroups of $\mathbb{Q}$. Then G is strongly indecomposable if and only if it is a Richman–Butler group.*

Proof. If G is a Richman–Butler group, then it is strongly indecomposable by Theorem 3.8 and the proof of Theorem 2.8. Conversely, assume that G is strongly indecomposable and let $\mu_i = \text{type}\,(A_i)$ for each $i \in \Omega_n$. Then, by Theorem 3.8, each μ_i is in $\Delta(G)$. Now, for a given $k \in \Omega_n$, let P_μ, Q_μ, I_μ and J_μ be as in Corollary 3.3 with $\mu = \mu_k$. Without loss of generality, we may suppose that $k \in J_\mu$. Since $\mu_i \wedge \mu_k \leqslant \mu_k = \mu$ for all $i \in \Omega_n$, $J_\mu = \{k\}$ and consequently $P_k = P_\mu$ possesses the properties required by Definition 2.7.

For $x = (a_1, ..., a_n) \in G = G(A_1, ..., A_n)$, we let $\text{supp}\,(\theta_G(x))$ denote the support of $\theta_G(x)$ in A_G, that is, $\text{supp}(\theta_G(x))$ consists of those $\mu \in \Delta(G)$ such that $\pi_\mu(\theta_G(x)) = \sum_{i \in I_\mu} a_i \neq 0$. We say that $\theta_G(x)$ is *uniform* provided there is a $d \in \mathbb{Q}$ such that the only values assumed by the $\pi_\mu(\theta_G(x))$'s are d, $-d$, and 0. Notice that if g_{ij} is a nonzero element of $G_{ij} = G \cap (A_i \oplus A_j)$, then $\theta_G(g_{ij})$ is uniform. Indeed, in this case, $g_{ij} = (a_1, ..., a_n)$ here $a_k = 0$ for $k \neq i$ and $k \neq j$, and $a_j = -a_i \neq 0$. Then from the definition of θ_G, the only possible values for $\pi_\mu \theta_G(g_{ij})$ are 0, a_i, and $-a_i$.

The lemma (appearing in somewhat different guise as Proposition 1.5 in [AV5]) that we shall state next is the crux of our proof of he isomorphism theorem for Richman–Butler groups. Moreover, the results accumulated thus far in this section are precisely the tools required to carry out the inductive step essential to the proof of the lemma.

LEMMA 3.10. *Suppose that* $G = G(A_1, ..., A_n)$ *is strongly indecomposable and quasi-isomorphic to* $G' = G(A'_1, ..., A'_n)$ *where both* $(A_1, ..., A_n)$ *and* $(A'_1, ..., A'_n)$ *are trimmed n-tuples of subgroups of* $\mathbb{Q}$. *If* g_{ij} *is a nonzero element of* G_{ij} *and if* x_{ij} *is an element of* G' *such that* $\operatorname{supp}(\theta_{G'}(x_{ij})) = \operatorname{supp}(\theta_G(g_{ij}))$, *then* $\theta_{G'}(x_{ij})$ *is uniform.*

Proof. Notice that the hypotheses imply that $\Delta(G) = \Delta(G')$ and, in view of Theorem 3.8, $\Delta(G)$ contains each $\mu_i = \text{type}(A_i)$ and each $\mu'_i = \text{type}(A'_i)$. The proof is by induction on n. First we dispose of the trivial case where $S_{ij} = \operatorname{supp}(\theta_G(g_{ij}))$ consists of only the two elements μ_i and μ_j. Then if $x_{ij} = (a'_1, ..., a'_n)$, the definition of $\theta_{G'}$ and the fact that each μ'_k lies in $\Delta(G)$ quickly lead us to the conclusion that there must be two distinct indices $k, l \in \Omega_n$ such that $a'_i = 0$ if $i \neq k$ and $i \neq l$, and $a'_l = -a'_k \neq 0$. But under these circumstances, $\theta_{G'}(x_{ij})$ is obviously uniform.

Now consider the general case where $|S_{ij}| \geqslant 3$ and let μ be an arbitrary element of S_{ij} distinct from μ_i and μ_j. Clearly from the definition of g_{ij}, $\mu \neq \mu_k$ for all $k \in \Omega_n$. The uniformity of $\theta_{G'}(x_{ij})$ will follow if we can show that

$$|\pi_{\mu_i}\theta_{G'}(x_{ij})| = |\pi_\mu \theta_{G'}(x_{ij})| = |\pi_{\mu_j}\theta_{G'}(x_{ij})|, \qquad (*)$$

where in this instance the vertical bars indicate absolute values. By Corollary 3.3, we can write $G[\mu] = G(A_{i_1}, ..., A_{i_s}) \oplus G(A_{j_1}, ..., A_{j_t})$ when $s \geqslant 2$ and $t \geqslant 2$. Moreover, since $\pi_\mu \theta_G(g_{ij}) \neq 0$ we may assume without loss of generality that $i \in I_\mu = \{i_1, ..., i_s\}$ and $j \in J_\mu = \{j_1, ..., j_t\}$. To establish the first equation in $(*)$, let $\sigma = \bigvee \{\mu_{i_k} : k = 1, ..., s\}$. Then, by Theorem 3.7, $\bar{G} = G(A_{i_1}, ..., A_{i_s}, A_\mu)$ is a strongly indecomposable group isomorphic to $G/G[\sigma]$. But since G and G' are quasi-siomorphic, so are $G/G[\sigma]$ and $G'/G'[\sigma]$, that is, $G'/G'[\sigma]$ is also strongly indecomposable. Now let $\rho: G \to \bar{G}$ and $\rho': G' \to \bar{G}'$ be the canonical maps as in Theorem 3.1. Note that $\bar{g}_{i\mu} = \rho(g_{ij}) \in \bar{G}_{i\mu} = \bar{G} \cap (A_i \oplus A_\mu)$ and furthermore, by Corollary 3.6, the τ-components of $\theta_{\bar{G}}(\bar{g}_{i\mu})$ and $\theta_{\bar{G}'}(\rho'(x_{ij}))$ are the same as the τ-components of $\theta_G(g_{ij})$ and of $\theta_{G'}(x_{ij})$, respectively, whenever $\tau \in \Delta(\bar{G}) = \Delta(\bar{G}')$. Thus the hypotheses of the lemma hold for $\bar{G}$, $\bar{G}'$, $\rho(g_{ij})$ and $\rho'(x_{ij})$. Observing that $s + 1 < n$, our induction hypotheses implies that $\theta_{\bar{G}'}(\rho'(x_{ij}))$ is uniform. Therefore the first equality in $(*)$ holds. Obviously the second equality of $(*)$ can be obtained by a similar argument with σ replaced by $\sigma' = \bigvee \{\mu_{jk} : k = 1, ..., t\}$.

For the remainder of this section, we assume that $G = G(A_1, ..., A_n)$ is a Richman–Butler group with $(A_1, ..., A_n)$ a trimmed n-tuple of subgroups of $\mathbb{Q}$. Therefore $\{\mu_1, ..., \mu_n\} \subseteq \Delta(G)$ where $\mu_i = \text{type}(A_i)$ for each $i \in \Omega_n$. Furthermore, since Butler groups have only finitely many cotypes [AV1, Theorem 1.7], $A_G = \bigoplus_{\mu \in \Delta(G)} A_\mu$ is a finite rank group. Notice also that $\theta_G : G \to A_G$ is a pure imbedding since this map followed by the projection onto $\bigoplus_{i \in \Omega_n} A_{\mu_i}$ is, in essence, the imbedding of G in $A_1 \oplus \cdots \oplus A_n$. The *extended type vector* of G is computed in A_G from the height vector $|\mathbf{c}| = (\alpha_\mu)_{\mu \in \Delta(G)}$ where c is an arbitrary nonzero element of $\bigcap_{\mu \in \Delta(G)} A_\mu = \bigcap_{i \in \Omega_n} A_i$ and α_μ denotes the height of c in A_μ. Clearly, $|\mathbf{c}|$ includes the components of the height vector of c computed, as in Section 2, within $A_1 \oplus \cdots \oplus A_n$. As we shall see in the proof of Theorem 3.11 below, the extended type vector is not only an isomorphism invariant for the Richman–Butler group G, but it is also unique. Moreover, since the components of $|\mathbf{c}|$ are indexed by $\Delta(G)$, if G and G' are Richman–Butler groups with the same extended type vector, then $\Delta(G) = \Delta(G')$.

Another ingredient needed in the formulation of the uniqueness theorem is the notion of the generalized radicals $G[M] = \bigcap_{\mu \in M} G[\mu]$ where M is an arbitrary set of types. If $M \subseteq \Delta(G)$, then obviously $x \in G[M]$ if and only if $M \cap \text{supp}(\theta_G(x)) = \varnothing$. A crucial fact is that $G_{ij} = G[M_{ij}]$ where $M_{ij} = \Delta(G) \setminus \text{supp}(\theta_G(g_{ij})$ for any nonzero g_{ij} in G_{ij}. That $G_{ij} \subseteq G[M_{ij}]$ follows from the general observation above; and the reverse inclusion is a consequence of the fact that if $x \in G \setminus G_{ij}$, then $\mu_k \in M_{ij} \cap \text{supp}(\theta_G(x))$ for some k in $\Omega_n \setminus \{i, j\}$. Yet another important observation concerning the Richman–Butler group $G = G(A_1, ..., A_n)$ is the fact that

$$G[M_{ij} \cap M_{jk}] = \langle G_{ij}, G_{jk} \rangle_*$$

whenever i, j, k are distinct indices in Ω_n. The inclusion $\langle G_{ij}, G_{jk} \rangle_* = \langle G[M_{ij}], G[M_{jk}] \rangle_* \subseteq G[M_{ij} \cap M_{jk}]$ is immediate and the reverse inclusion follows from the fact that $\langle G_{ij}, G_{jk} \rangle_* = G[M]$ for a certain subset M of $M_{ij} \cap M_{jk}$. Indeed, since $G[\mu_l] = G \cap \bigoplus_{i \neq l} A_i$ for each $l \in \Omega_n$, it is clear that $\langle G_{ij}, G_{jk} \rangle_* = G \cap (A_i \oplus A_j \oplus A_k)$ is equal to $\bigcap_{\mu \in M} G[\mu]$ where $M = \{\mu_l : l \in \Omega_n \setminus \{i, j, k\}\}$.

We now formulate the uniqueness theorem for Richman–Butler groups, which also appears as Theorem 1.6 in [AV5].

THEOREM 3.11. *Let $G = G(A_1, ..., A_n)$ and $G' = G(A'_1, ..., A'_n)$ be Richman–Butler groups where $(A_1, ..., A_n)$ and $(A'_1, ..., A'_n)$ are trimmed n-tuples of subgroups of $\mathbb{Q}$. Then G and G' are isomorphic if and only if they have the same extended type vector and* $\text{rank}(G[M]) = \text{rank}(G'[M])$ *for all subsets M of $\Delta(G) = \Delta(G')$.*

We first show that the conditions of Theorem 3.11 are necessary for the Richman–Butler groups $G = G(A_1, ..., A_n)$ and $G' = G(A'_1, ..., A'_n)$ to be isomorphic. Assume then that there exists an isomorphism $\phi\colon G \to G'$. Since obviously ϕ maps $G[M]$ onto $G'[M]$ for each set of types M, the condition $\operatorname{rank}(G[M]) = \operatorname{rank}(G'[M])$ is certainly satisfied. In particular, ϕ maps $G[\mu]$ onto $G'[\mu]$ for all $\mu \in \Delta = \Delta(G)$. Recalling that, for each $\mu \in \Delta$, $\pi_\mu \theta_G\colon G \to A_\mu$ and $\pi_\mu \theta_{G'}\colon G' \to A'_\mu$ are epimorphisms with kernels $G[\mu]$ and $G'[\mu]$, respectively, we see that there are induced isomorphisms $\phi_\mu\colon A_\mu \to A'_\mu$ such that the diagram

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "\pi_\mu \theta_G"] & G' \arrow[d, "\pi_\mu \theta_{G'}"] \\ A_\mu \arrow[r, "\phi_\mu"] & A'_\mu \end{tikzcd}$$

is commutative for each $\mu \in \Delta$. It then follows that $\psi = \bigoplus_{\mu \in \Delta} \phi_\mu$ is an isomorphism of A_G onto $A_{G'}$ such that $\psi\theta_G = \theta_{G'}\phi$. We now wish to use these observations to show that G and G' have the same extended type vector. Towards this end, select an arbitrary nonzero c in $\bigcap_{\mu \in \Delta} A_\mu$ and let $c' = \phi_{\mu_1}(c)$. The identity of the extended type vectors will follow once we show that $c' \in \bigcap_{\mu \in \Delta} A'_\mu$ and that, for each $\mu \in \Delta$, the height of c' in A'_μ is the same as the height of c in A_μ. The latter clearly holds for $\mu = \mu_1$ and also, by choice, $c' \in A'_{\mu_1}$. So let μ be any element of Δ distinct from μ_1. Next observe that there is a $j \geqslant 2$ such that $\mu \in S_{1j} = \Delta \setminus M_{1j}$ and that there is a g_{1j} in G_{1j} such that each nonzero component of $\theta_G(g_{1j})$ is c or $-c$. Now let $x_{1j} = \phi(g_{1j})$ and observe that $\operatorname{supp}(\theta_{G'}(x_{1j})) = \operatorname{supp}(\theta_G(g_{1j})) = S_{ij}$ since $\psi\theta_G(g_{1j}) = \theta_{G'}(x_{1j})$. Also notice that $\pi_{\mu_1}\theta_{G'}(x_{1j}) = \phi_{\mu_1}\pi_{\mu_1}\theta_G(g_{1j}) = \phi_{\mu_1}(c) = c'$ since we may assume that c is the μ_1-component of $\theta_G(g_{1j})$. Then, by Lemma 3.10, $\theta_{G'}(x_{1j})$ is uniform and hence its μ-component is either c' or $-c'$. But, of course, the identity $\phi_\mu \pi_\mu \theta_G = \pi_\mu \theta_{G'} \phi$ tells us that ϕ_μ maps the μ-component of $\theta_G(g_{1j})$ to the μ-component of $\theta_{G'}(x_{1j})$, and hence $\phi_\mu(c) = \pm c'$. Therefore $c' \in A'_\mu$ and the height of c' in A'_μ is the same as the height of c in A_μ. This completes the proof that the conditions of Theorem 3.11 are necessary for the Richman–Butler groups G and G' to be isomorphic.

In broad outline, the proof of the sufficiency of the conditions in Theorem 3.11 is effected by choosing for each $g_{ij} \in G_{ij}$ an appropriate $x_{ij} \in G'$ such that the desired isomorphism $\phi\colon G \to G'$ can be constructed by taking $\phi(g_{ij}) = x_{ij}$. In finer detail, however, we require A_G and $A_{G'}$ to guide our choice of the x_{ij}'s and we actually construct an isomorphism $\psi\colon A_G \to A_{G'}$ that maps $\theta_G(G) \cong G$ onto $\theta_{G'}(G') \cong G'$. Since $g_{ij} \in G_{ij} = G[M_{ij}]$, we must, of course, choose x_{ij} in $G'[M_{ij}]$. First let us observe that if x_{ij} is an arbitrary nonzero element in $G'[M_{ij}]$, then $\operatorname{supp}(\theta_{G'}(x_{ij})) = S_{ij}$

where $S_{ij} = \operatorname{supp}(\theta_G(g_{ij}))$. That $\operatorname{supp}(\theta_{G'}(x_{ij})) \subseteq S_{ij}$ follows from the fact that $M_{ij} \cap \operatorname{supp}(\theta_{G'}(x_{ij})) = \varnothing$. If this inclusion were proper, then there would be some $\mu \in S_{ij}$ such that $x_{ij} \in G'[M_{ij}] \cap G'[\mu]$. But since $g_{ij} \notin [\mu]$ and $G[M_{ij}] = G_{ij}$ has rank 1, $G[M_{ij}] \cap G[\mu] = 0$. This, however, contradicts the hypotheses of Theorem 3.11 if we take $M = M_{ij} \cup \{\mu\}$.

Naturally, we must select the x_{ij}'s to satisfy some conditions beyond $\operatorname{supp}(\theta_{G'}(x_{ij})) = S_{ij}$. Furthermore, it should be evident that Lemma 3.10 will need to be invoked in order to construct the desired isomorphism $\psi: A_G \to A_{G'}$. But to apply that lemma we need G and G' to be quasi-isomorphic, a requirement that does not appear to follow immediately from the given hypotheses. It does, however, turn out that Richman–Butler groups G and G' are quasi-isomorphic if and only if the following condition is satisfied:

$$\Delta(G) = \Delta(G') \quad \text{and} \quad \operatorname{rank}(G[M]) = \operatorname{rank}(G'[M]) \quad \text{for all} \quad M \subseteq \Delta(G). \tag{$**$}$$

This result is, in fact, proved in [AV3]. But, in the present context, we can give a simpler proof; and indeed with the the tools now available the proofs of quasi-isomorphism and isomorphism are but trivial modifications of one another.

Let us then assume, for the moment, that G and G' are Richman–Butler groups satisfying condition $(**)$ of the preceding paragraph. We shall write Δ for $\Delta(G) = \Delta(G')$. Since $A_\mu \cong A'_\mu$ for all $\mu \in \Delta$, it is evident that we can choose elements $x_{1j} \in G'[M_{1j}]$ for $2 \leqslant j \leqslant n$ such that the following two conditions are satisfied:

(1) For each $\mu \in S_{1j}$, the height of $\pi_\mu \theta_{G'}(x_{1j})$ in A'_μ is not less than the height of $\pi_\mu \theta_G(g_{1j})$ in A_μ.

(2) $\pi_{\mu_1} \theta_{G'}(x_{1j}) = \pi_{\mu_1} \theta_{G'}(x_{1k})$ for $2 \leqslant j < k \leqslant n$.

Then clearly, for each j, there is a monomorphism $\psi_j: \bigoplus_{\mu \in S_{1j}} A_\mu \to \bigoplus_{\mu \in S_{1j}} A'_\mu$ such that

(3) $\psi_j(\theta_G(g_{1j})) = \theta_{G'}(x_{1j})$;

(4) $\psi_j(A_\mu) \subseteq A'_\mu$ for all $\mu \in S_{1j}$.

Since $\Delta = \bigcup_{j=2}^n S_{1j}$, there will exist a monomorphism $\psi: A_G \to A_{G'}$ extending all the ψ_j's provided the following compatibility condition is satisfied:

(5) $\pi_\mu \theta_{G'}(x_{1j}) = \pi_\mu \theta_{G'}(x_{1k})$ for all $\mu \in S_{1j} \cap S_{1k}$.

We shall now show that if (5) should fail, then $(**)$ would be contradicted. Towards this end, let $N = M_{1j} \cap M_{1k}$ where $2 \leqslant j < k \leqslant n$ and recall our earlier observation that $G[N]=\langle G_{1j}, G_{1k}\rangle_*$. By (**) $\operatorname{rank}(G'[N])=\operatorname{rank}(G[N])$ and consequently $G'[N]=\langle G'[M_{1j}], G'[M_{1k}]\rangle_*$. Then, by (2), $G'[N]\cap G'[\mu_1]=\langle x_{1j}-x_{1k}\rangle_*$ and, similarly, $G[N]\cap G[\mu_1]=\langle g_{1j}-g_{1k}\rangle_*$. Now suppose (5) fails and let $\mu\in S_{1j}\cap S_{1k}$ be such that $\pi_\mu\theta_{G'}(x_{1j})\neq\pi_\mu\theta_{G'}(x_{1k})$. Then $x_{1j}-x_{1k}\notin G'[\mu]$ and therefore $G'[N]\cap G'[\mu_1]\cap G'[\mu]=0$. But since $g_{1j}-g_{1k}$ necessarily lies in $G[\mu]$ when $\mu\in S_{1j}\cap S_{1k}$, $G[N]\cap G[\mu_1]\cap G[\mu]=\langle g_{1j}-g_{1k}\rangle_*\neq 0$. Therefore (**) is contradicted for $M=N\cup\{\mu_1,\mu\}$. Thus (5) is satisfies and the desired monomorphism $\psi\colon A_G\to A_{G'}$ exists. Then because $\theta_G(G)$ and $\theta_{G'}(G')$ are pure in A_G and $A_{G'}$, respectively, condition (3) insures that ψ maps $\theta_G(G)$ into $\theta_{G'}(G')$. Hence we see that there is a monomorphism from G to G' and, by the symmetry of (**), there is also a monomorphism from G' to G, that is, G and G' are quasi-isomorphic.

Returning to the hypotheses of Theorem 3.11, we now assume that G and G' are Richman–Butler groups having the same extended type vector and also satisfying condition (**) above. Then we can select nonzero elements c in $\bigcap_{\mu\in\Delta}A_\mu$ and c' in $\bigcap_{\mu\in\Delta}A'_\mu$ such that, for each μ, c and c' have the same height in A_μ and A'_μ, respectively. We next choose $g_{1j}\in G_{1j}$, for $2\leqslant j\leqslant n$, such that each nonzero component of $\theta_G(g_{1j})$ is either c or $-c$. By Lemma 3.10, we can also select $x_{1j}\in G'[M_{1j}]$ such that each nonzero component of $\theta_{G'}(x_{1j})$ is either c' or $-c'$. Then obviously we have, for each j, an isomorphism $\psi_j\colon\bigoplus_{\mu\in S_{1j}}A_\mu\to\bigoplus_{\mu\in S_{1j}}A'_\mu$ which satisfies condition (3) of the preceding paragraph and also

(4′) $\quad\psi_j(A_\mu)=A'_\mu$ for all $\mu\in S_{1j}$.

Indeed we can choose ψ_j so that $(\psi_j|A_\mu)(c)=\pm c'$ for all $\mu\in S_{1j}$. Since we may also assume that the x_{1j}'s are chosen so as to satisfy condition (2) of the preceding paragraph, the fact that $\operatorname{rank}(G[M])=\operatorname{rank}(G'[M])$ for all $M\subseteq\Delta$ implies, as before, that the compatibility condition (5) is once again satisfied. Therefore there exists an isomorphism $\psi\colon A_G\to A_{G'}$ extending each of the ψ_j's above. Finally, observe that ψ maps $\theta_G(G)$ onto $\theta_{G'}(G')$ because of condition (3) and the purity of these subgroups. This completes the proof that the conditions of Theorem 3.11 are sufficient to insure that the Richman–Butler groups G and G' are isomorphic.

COROLLARY 3.12 [AV3]. *The Richman–Butler groups $G=G(A_1,\ldots,A_n)$ and $G'=G(A'_1,\ldots,A'_n)$ are quasi-isomorphic if and only if $\Delta(G)=\Delta(G')$ and* $\operatorname{rank}(G[M])=\operatorname{rank}(G'[M])$ *for all $M\subseteq\Delta(G)$.*

Proof. The sufficiency of the given conditions was established in the proof of Theorem 3.11. The converse, however, is essentially trivial since $\Delta(G)$ and the ranks of the various $G[M]$'s are readily seen to be quasi-isomorphic invariants of G.

As we mentioned in the introduction, independently Arnold and Vinsonhaler [AV5] have also proved a uniqueness theorem for Richman–Butler groups similar to Theorem 3.11; however their proof relies heavily on the quasi-isomorphism theorem of [AV3].

4. Butler Groups as Homomorphic Images

As we remarked earlier, one of the fundamental properties of Butler groups (= pure subgroups of finite rank completely decomposable groups) is that they are also precisely those torsion-free groups that are homomorphic images of completely decomposable groups of finite rank [B]. Moreover, if G is a Butler group, there always exists a *balanced exact sequence*

$$0 \to K \to A \to G \to 0$$

where A is a finite rank completely decomposable group [AV1, Theorem 1.2]. In this section, we shall investigate Butler groups G using this representation, though eventually we shall relax the hypothesis that K is balanced in A.

THEOREM 4.1. *Let* $0 \to K \to A \to G \to 0$ *and* $0 \to K' \to A \to G' \to 0$ *be balanced exact sequences where* A *is a finite rank completely decomposable group with incomparable homogeneous components. Then the Butler groups* G *and* G' *are* B_0*-groups where* $G \cong G'$ *if and only if* K *and* K' *are equivalent subgroups of* A.

Proof. By Proposition 1.7(i), K and K' are weakly $*$-pure in A and therefore, by part (ii) of the same proposition, both G and G' are B_0-groups. Clearly G and G' will be isomorphic if K and K' are equivalent subgroups of A. Conversely, assume that $G \cong G'$. Then, since the short exact sequences are balanced, an isomorphism from G to G' can be construed as an isomorphism $\phi\colon A/K \to A/K'$ that respects heights. Thus Theorem 2.1 is applicable to complete the proof.

A fundamental fact, first proved in [B], is that if G is a Butler group and if σ is a type, then $G(\sigma) = G_\sigma \oplus G(\sigma^*)_*$ where G_σ is a homogeneous, completely decomposable group of type σ and, of course, $G(\sigma^*)_*$ denotes the pure closure of $G(\sigma^*)$. We shall refer to rank $G_\sigma = \operatorname{rank}(G(\sigma)/G(\sigma^*)_*)$ as the σth Baer invariant of the Butler group, G, and we let $P_G = \{\sigma : G(\sigma) \neq G(\sigma^*)_*\}$. If G is a B_0-groups, then it is easily seen that $G = \sum_{\sigma \in P_G} G_\sigma$ and that the canonical map $\bigoplus_{\sigma \in P_G} G_\sigma \to G$ is a balanced epimorphism [A, Theorem 2.2].

THEOREM 4.2. *There is a natural bijection between the isomorphism classes of B_0-groups G with prescribed Baer invariants manifested by $A_\sigma \simeq G(\sigma)/G(\sigma^*)$ and the equivalence classes of balanced subgroups K of the fixed group $A = \bigoplus A_\sigma$ that satisfy $K \cap A(\sigma) = K \cap A(\sigma^*)$.*

Proof. Suppose G is a B_0-group and let K be the kernel of the canonical map $\bigoplus_{\sigma \in P_G} G_\sigma \to G$. Then K is not only a balanced subgroup of $A = \bigoplus_{\sigma \in P_G} G_\sigma$, but in fact $K(\sigma) \subseteq A(\sigma^*)$ for all types $\sigma \in P_G$. Indeed if $\tau \in P_G$ and $a = (..., g_\sigma, ...) \in K(\tau)$, then clearly $g_\sigma = 0$ for $\sigma \ngeq \tau$ and also $g_\tau = -\sum_{\sigma > \tau} g_\sigma \in G_\tau \cap G(\tau^*) = 0$. An application of Corollary 1.6 yields the desired conclusion.

In general, of course, the determination of the equivalence classes of balanced subgroups of $A = \bigoplus A_\sigma$ appears to be only slightly more tractable than that of the isomorphism classes of B_0-groups with specified Baer invariants. Nonetheless, if we are willing to restrict attention to sufficiently specialized classes of B_0-groups., then definitive results are available. If $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ where the A_i's are the homogeneous components of the completely decomposable group A and if $a = (a_1, a_2, ..., a_n)$ is an arbitrary element of A, then we can associate with a the height vector $(\alpha_1, \alpha_2, ..., \alpha_n)$ where $\alpha_i = |\alpha_i|$ for $i = 1, 2, ..., n$. Clearly all elements of $\langle a \rangle_*$ have the same associated type vector.

THEOREM 4.3. *If G is a B_0-group such that the types in P_G are pairwise incomparable and if* $\operatorname{rank}(G) + 1 = \sum_{\sigma \in P_G} \operatorname{rank}(G(\sigma)/G(\sigma^*))$, *then G is determined up to isomorphism by its Baer invariants and the type vector associated with the kernel of the natural map $\bigoplus_{\sigma \in P_G} G_\sigma \to G$. Moreover, the latter is an invariant of the B_0-group G.*

Proof. Given exact sequences $0 \to K \to A \to G \to 0$ and $0 \to K' \to A' \to G' \to 0$ where $\operatorname{rank}(K) = \operatorname{rank}(K') = 1$ and the groups A and A' are isomorphic and completely decomposable, then obviously K and K' having the same type vector suffices to yield an isomorphism from A to A' that maps K onto K'. In particular, $G \cong G'$ under these circumstances. Furthermore, if $A = \bigoplus_{\sigma \in P_G} G_\sigma$, $A' = \bigoplus_{\sigma \in P_{G'}} G'_\sigma$ and the B_0-groups G and G' are isomorphic, then Theorem 4.2 insures the existence of an isomorphism from A to A' that maps K onto K'. But when the types in $P_G = P_{G'}$ are pairwise incomparable, any isomorphism between A and A' must preserve height vectors.

In reference to Theorem 4.3, there are two salient observations that merit comment. Firstly, it is generally the case that rank 1 subgroups are not balanced; secondly, the proof of the first part of the theorem does not actually require the rank 1 subgroups to be balanced. With regard to the first point, suppose $a = (a_1, ..., a_n)$ is an element of the completely decomposable group $A=A_1\oplus\cdots\oplus A_n$ where the A_i's are the homogeneous components of A and, for each i in $\Omega_n=\{1,...,n\}$, μ_i is the type of A_i. If $a_i\neq 0$ for each $i\in\Omega_n$, then $\langle a\rangle_*$ is a balanced subgroup of A only if the following highly restrictive condition is satisfied: For each partition I, J of Ω_n, the types $\bigwedge_{i\in I}\mu_i$ and $\bigwedge_{j\in J}\mu_j$ are comparable. Both observations, of course, suggest that we henceforth relax the assumption that the rank 1 kernels be balanced.

Pursuing this suggestion, we quickly see that it suffices to limit ourselves to a construction that is dual to the one that played such a prominent role in the two preceding sections. For any n-tuple $(A_1,...,A_n)$ of subgroups of $\mathbb{Q}$, we let $G[A_1,...,A_n]$ denote the cokernel of the diagonal map $\delta_A:\bigcap_{i\leqslant n}A_i\to\bigoplus_{i\leqslant n}A_i$ given by $a\to(a,...,a)$. In this context, the type vector of Theorem 4.3 is computed in precisely the same manner as the type vector of Section 2, namely, it is the type vector associated with the height vector $|\mathbf{c}|$ formed from any nonzero c in $\bigcap_{i\leqslant n}A_i$. Furthermore, the following dual of Theorem 2.4 is immediate from the first part of the proof of Theorem 4.3.

THEOREM 4.4. *Let $(A_1,...,A_n)$ and $(B_1,...,B_n)$ be two n-tuples of nonzero subgroups of $\mathbb{Q}$ and consider the exact sequences*

$$0\longrightarrow\bigcap_{i\leqslant n}A_i\xrightarrow{\delta_A}\bigoplus_{i\leqslant n}A_i\longrightarrow G[A_1,...,A_n]\longrightarrow 0\qquad\text{(E)}$$

and

$$0\longrightarrow\bigcap_{i\leqslant n}B_i\xrightarrow{\delta_B}\bigoplus_{i\leqslant n}B_i\longrightarrow G[B_1,...,B_n]\longrightarrow 0\qquad\text{(E}'))$$

If (E) *and* (E′) *have the same type vector, then they are equivalent exact sequences through isomorphism $A_i\to B_i$. In particular, $G[A_1,...,A_n]$ and $G[B_1,...,B_n]$ are isomorphic.*

Recalling various dual results for Butler groups that have appeared in the literature, most notably in [B] and [AV1], a comparison of Theorem 2.4 and Theorem 4.4 suggests that one might be able to dualize all the results in Section 3 to the setting of groups of the special form $G[A_1,...,A_n]$. For instance, the kernels K_{ij} of the canonical maps $G[A_1,...,A_n]\to A_i\oplus A_j$ (induced by the maps $(a_1,...,a_n)\to a_i-a_j$) would play the same role as the subgroups G_{ij} of $G(A_1,...,A_n)$ introduced in Lemma 2.6. In fact, using [AV1, Theorem 1.4], one easily dualizes that lemma to obtain $G(\sigma)=\bigcap\{K_{ij}:\sigma\nleq\mu_i\vee\mu_j\}$ for each type σ. This observation, of course, compels one to introduce co-Richman–Butler groups by simply imitating Definition 2.7, that is, one merely replaces the inequalities $\mu_i \wedge \mu_j \nleq \mu_k$ in that definition by the corresponding inequalities $\mu_k \nleq \mu_i \vee \mu_j$ and calls an n-tuple $(A_1, ..., A_n)$ contrimmed provided $A_i = A_i + \bigcap_{j \neq i} A_j$ for each i. The further required modifications of entities introduced in Section 3 should be fairly evident: The types μ of elements in $G = G[A_1, ..., A_n]$ such that rank $G(\mu) = 1$ yield a distinguished family of types $\nabla(G)$ and there is a canonical epimorphism $A^G \to G$ where A^G is the direct sum of all the rank 1 groups $A_\mu = (\bigcap_{i \in I_\mu} A_i) + (\bigcap_{j \in J_\mu} A_j)$ as μ ranges over $\nabla(G)$. Finally, the dual version of Theorem 3.11 can be established with the extended type vector for $G = G[A_1, ..., A_n]$ being computed in A^G and the other required invariants being the finite cardinals rank $G(M) = \text{rank}\,(\sum_{\mu \in M} G(\mu))$ with $M \subseteq \nabla(G)$. The tedious details of such a proof, however, can now be avoided by applying the definitive formalization of *Butler duality* that has been developed in [AV4]. Indeed in that paper, Arnodl and Vinsonhaler utilize their general theory to dualize results in [AV3] thereby obtaining quasi-isomorphism invariants for strongly indecomposable $G[A_1, ..., A_n]$'s. Then, in [AV5], exploiting a variation of the extended type vector, they fine tune their quasi-isomorphism theorems to obtain both our Theorem 3.1 and its dual.

There have been other noteworthy developments in the theory of Butler groups subsequent to the original version of the present paper. In [FM] there appears an alternative approach to the quasi-isomorphism problem for co-Richman–Butler groups $G = G[A_1, ..., A_n]$. When $H = G[B_1, ..., B_n]$ is another such group with $\nabla(H) = \nabla(G)$, then Fuchs and Metelli associate an $n \times n$ $\{0, 1\}$-matrix ε such that G and H are quasi-isomorphic if and only if ε is admissible in the following sense: If ε_k is the matrix obtained from ε by replacing all 0's in the k-th column by 1's, then det $\varepsilon_k \neq 0$ for $k = 1, ..., n$. Further light has been shed on the nature of these Fuchs–Metelli matrices by as yet unpublished investigations that show them to be matrices with entries from the field $\mathbb{Z}_2 = \{0, 1\}$ associated with linear transformations implementing an equivalence between vector space representations of the posets consisting of the types realized by the nonzero elements of G and H. Moreover, there is now strong evidence that all groups (not just the strongly indecomposable ones) of the special forms $G(A_1, ..., A_n)$ and $G[A_1, ..., A_n]$ are classifiable up to quasi-isomorphism by the invariants rank $G[M]$ and rank $G(M)$, respectively. On the other hand, examples obtained independently by the authors and David Arnold show that these invariants are not adequate to distinguish the quasi-isomorphism classes of corank 2 Butler groups.

REFERENCES

[A] D. ARNOLD, Pure subgroups of finite rank completely decomposable groups, *in* "Lecture Notes in Mathematics," Vol. 874, pp. 1–31, Springer-Verlag, New York, 1981.

[AV1] D. ARNOLD AND C. VINSONHALER, Pure subgroups of finite rank completely decomposable groups, II, *in* "Lecture Notes in Mathematics," Vol. 1006, pp. 97–143, Springer-Verlag, New York, 1983.

[AV2] D. ARNOLD AND C. VINSONHALER, Representing graphs for a class of torsion-free groups, *in* "Abelian Group Theory," pp. 309–332, Gordon & Breach, New York, 1987.

[AV3] D. ARNOLD AND C. VINSONHALER, Invariants for a class of torsion-free abelian groups, *Proc. Amer. Math. Soc.* **105** (1989), 293–300.

[AV4] D. ARNOLD AND C. VINSONHALER, Duality and invariants for Butler groups, *Pacific J. Math.* **148** (1991), 1–10.

[AV5] D. ARNOLD AND C. VINSONHALER, Isomorphism invariants for abelian groups, *Trans. Amer. Math. Soc.* **330** (1992), 711–724.

[B] M. C. R. BUTLER, A class of torsion-free abelian groups of finite rank, *Proc. London Math. Soc.* **15** (1965), 680–698.

[DR] M. DUGAS AND K. RANGASWAMY, On torsion-free abelian k-groups, *Proc. Amer. Math. Soc.* **99** (1987), 403–408.

[F] L. FUCHS, "Infinite Abelian Groups," Vol. II, Academic Press, New York, 1973.

[FM] L. FUCHS AND C. METELLI, On a class of Butler groups, *Manuscripte Math.* **71** (1991), 1–28.

[H1] P. HILL, The classification problem, *in* "Abelian Groups and Modules" (Proceedings of the Udine Conference), pp. 1–16, Springer-Verlag, New York, 1984.

[H2] P. HILL, On the structure of abelian p-groups, *Trans. Amer. Math. Soc.* **288** (1985), 505–525.

[HM1] P. HILL AND C. MEGIBBEN, On the theory and classification of abelian p-groups, *Math. Z.* **190** (1985), 17–38.

[HM2] P. HILL AND C. MEGIBBEN, Torsion free groups, *Trans. Amer. Math. Soc.* **295** (1986), 735–751.

[HM3] P. HILL AND C. MEGIBBEN, The local equivalence theorem, *Contemp. Math.* **87** (1989), 201–219.

[HM4] P. HILL AND C. MEGIBBEN, Equivalence theorems for torsion-free groups, *in* "Abelian Groups" (Proceedings of the 1991 Curaçao conference), pp. 181–191, Marcel Dekker, New York, 1993.

[J] B. JÓNSSON, On direct decompositions of torsion-free abelian groups, *Math. Scand.* **7** (1959), 361–371.

[L] M. LANE, Isotype submodules of p-local balanced projective groups, *Trans. Amer. Math. Soc.* **301** (1987), 313–325.

[R] F. RICHMAN, An extension of the theory of completely decomposable torsion-free abelian groups, *Trans. Amer. Math. Soc.* **279** (1983), 175–185.

[W] R. WARFIELD, A classification theorem for abelian p-groups, *Trans. Amer. Math. Soc.* **210** (1975), 149–168.